Invariant Coordinate Selection (ICORS): A Robust Perspective on Independent Component Analysis (ICA)

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Abstract

An obvious method for generating robust measures of location for p-dimensional distributions is to simply apply robust univariate measures of location to each of the coordinates, e.g. the coordinatewise median. A drawback to this approach is that the resulting measure of multivariate location is not affine equivariate. If one could select the coordinates in an invariant manner, however, i.e. select p data dependent linear combinations of the variables which are invariant under nonsingular transformations of the variables, then applying coordinatewise measure of univariate location to the transformed variables and then back-transforming gives an affine equivariant measure of multivariate location. Affine covariant measure for the scatter matrix can also be generated using coordinatewise measures of scale.

To be more specific, let $Y = \{y_1, \ldots, y_n\}$ be a p-dimensional data set. Suppose we are able to define a nonsingular matrix $A(Y)$ such that the transformed p-dimensional data set $Z = A(Y)Y$ is invariant under nonsingular transformations of Y, i.e. $A(Y)Y = A(BY)BY$ for any nonsingular matrix B. If we then apply univariate measures of location and scale to each of the components of Z producing $\mu(Z) \in \Re^p$ and $\sigma(Z) \in \Re^p$ respectively, then affine equivariant measures of multivariate location and scatter can be defined by

$$\mu(Y) = A(Y)^{-1}\mu(Z) \quad \text{and} \quad \Sigma(Y) = A(Y)^{-1}D(\sigma^2(Z))(A(Y)')^{-1},$$

where $D(\cdot)$ is a diagonal matrix whose diagonal elements are given by its vector argument.

One method for generating such an invariant transformation is as follows. First compute two different affine covariate estimates of scatter for Y, say V_o and V_1, and then define $A(Y) = (a_1 \ldots, a_p)$ such that

$$V_o a_j = \gamma_j V_1 a_j \quad \text{for } j = 1, \ldots, p \ \text{ or equivalently,} \quad V_o A(Y) = V_1 A(Y)\Delta,$$

where $\Delta = D(\gamma_1, \ldots, \gamma_p)$. That is, $A(Y)$ are the principal component vectors of V_o, relative to the Mahalanobis inner product defined via V_1. The transformed variates $Z = A(Y)Y$ can be viewed as *affine invariant principal components*. In a personal communication, Hannu Oja has noted, that under certain conditions, the matrix $A(Y)^{-1}$ also represents a solution to the independent component analysis problem.

If the univariate location and scale statistics have breakdown point 1/2, then we note that the statistic $(\mu(Y), \Sigma(Y))$ breaks down only if the matrix $A(Y)$ approaches a singular matrix. The breakdown point of $(\mu(Y), \Sigma(Y))$ is shown to be at least as large as the larger of the breakdown points of V_o and V_1 and can be considerably larger. Under point mass contamination, the breakdown point is 1/2 regardless of the breakdown points of V_o and V_1. Measures of robustness and the concept of breakdown for the ICA problem are also discussed.

Finally, we note that one can produce affine invariant diagnostic plots by plotting the components of Z or by making pairwise scatter plots of the components of Z. In a sense, this can be regarded as *projection pursuit without the pursuit*. We give several examples which illustrates the utility of the proposed methods. This talk is based on the joint work with Hannu Oja of the University of Jyväskylä and Lutz Dümbgen of the University of Bern.